A SOLUTION TO THE COMPLETION PROBLEM OF QUASI-UNIFORM SPACES

ATHANASIOS ANDRIKOPOULOS   JOHN STABAKIS

Abstract. We give a new completion for the quasi-uniform spaces. We call the whole procedure *τ-completion* and the new space *τ-complement of the given*. The basic result is that every T_0 quasi-uniform space has a τ-completion. The τ-complement has some "crucial"properties, for instance, it coincides with the classical one in the case of uniform space or it extends the *Doitcinov's completion for the quiet spaces.* We use nets and from one point of view the technique of the construction may be considered as a combination of the *Mac Neille's cut* and of the completion of partially ordered sets via *directed subsets*.

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1. Introduction

It is our main purpose in this paper to give a standard construction for the completion of any T_0 quasi-uniform space. Császár was the first who developed a theory of completion for quasi uniform spaces [2]. Since then, the problem has been approached by several authors, but, up to now, none of the solutions proposed is able to give a satisfying completion theory for all quasi-uniform spaces. We must add that in two rather recent papers, [Bosangue and all, 1998] and [Kunzi and all, 2002], it is attempted to be given a definite resolution of the subject. In the first of them, starting from a Lawvere's idea [11] on a generalization of metric and ordered spaces, the authors give a completion of quasi-uniform spaces, rather according to the completion of some families of quasi-uniform spaces than to the completion of an ordered space. In the second, they give the completion of some families of generalized metric spaces considering it as the *Yoneda embedding*. However, although for some cases of quasi-uniform spaces a *standard construction* is possible, there is not a general method for any quasi-uniform space. Even simple examples, as, for instance, the topological ordered spaces, are not in general covered by the current constructions.

It is not difficult for one to understand that the problem is not an easy task. "*The problem of completing of quasi-uniform spaces is not a trivial one. The analogy with the uniform spaces provides a too small profit*"says Doitchinov in [6]. Sünderhauf in [14]: "*there are various attempts to define a notion of*

completeness and completion, but none of these is able to handle all spaces in a satisfying manner".

And later, the same author in [15]: "*as soon as the notion of Cauchy sequences* (*and filters*), *limits and completeness come into play, the situation becomes rather chaotic*". And others say such things in other papers.

From one point of view the problem of completing quasi-uniform spaces is due to the plethora of cases which one has to confront. This plethora is due to the fact that in a quasi-uniform spaces a "*natural*" Cauchy system does not converge in an also "*natural way*". Consider, for instance, the real line $\mathbb{R}$ with a topology τ, whose a subbase consists of the intervals $(\leftarrow, x]$, for any $x \in \mathbb{R}$. If $\mathcal{U}$ is a quasi-uniformity compatible with τ, then, for any $a \in \mathbb{R}$, any sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n < a$, ($\mathbb{N}$ the set of natural numbers), converges to a.

Such facts explaining why, amongst the many proposals, Doitchinov's D-completion (firstly in [4], [5], etc.) of the so-called "*quiet spaces*" was particularly promising. He introduced the idea of considering with every *Cauchy net*, a second net - the so called *conet of the former* - which couple of nets restricts the area where the "new point "must stand and so it is possible for one to give a completion, the so called D-*completion*, for the so-called *quiet spaces*. Unfortunately, although too many current examples of quasi-uniform spaces have D-completion, that was not a resolution: "*the quiet spaces constitute a very small class of spaces*" (cf. [12]). Fletcher, in one his reviewing, notes: "*D.Doitchinov showed that it is impossible to give a satisfactory theory of completion for the class of quasi-uniform spaces and introduced the class of quiet quasi-uniform spaces, a class comprising all uniform spaces for which a satisfactory theory of completion exists*".

The usual meaning of completeness is that every Cauchy system (sequence or net or filter), whatever such a system means, converges. So the definition of the Cauchy system is the first task. But such a definition is not always a natural one. In some cases the converging nets are not Cauchy nets, in some others they are Cauchy, but cease to stand as Cauchy when we withdraw the limit points from the space, in a third the completion has only one "*new point*" very artificially constructed and, finally, we "*complete*" with "*new points*" an already complete space. In this paper, we introduce a new notion, the τ-*cut* (from the Greek word $\tau o\mu\eta$ = cut). More precisely, we make use of pairs "net-conet" as in the D-completion, here as τ-*net* and τ-*conet*. The τ_p-nets are nets whose origin is due to Stoltenberg [13]. For more details in the subject see [9].

Our basic conclusion is the Theorem 28 which states that every T_0 quasi-uniform spaces has a τ-completion, that is a completion via the τ-cuts. The meaning of the τ-Cauchy net of the τ-completion is given in the Definitions 9 and 10. As we have already said there exists a large number of completions, among them the Deak's and the ones of Smyth's and Doitchinov's (there is a very large bibliography of Deak last papers; the paper [3] is among of them).

So, the paper is organized as follows: in the paragraphs 2 we give the basic definitions and structures for the τ-completion; paragraph 3 is referred to the

definition of a quasi-uniformity in the τ-completion and in the fourth paragraph we prove the τ-completeness of the new space we have constructed.

2. The τ-cut

Let us recall that a quasi-uniformity on a (nonempty) set X is a filter $\mathcal{U}$ on $X \times X$ which satisfies: (i) $\Delta(X) = \{(x,x) | x \in X\} \subseteq U$ for each $U \in \mathcal{U}$ and (ii) given $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. The elements of the filter $\mathcal{U}$ are called *entourages*. The pair $(X,\mathcal{U})$ is called a *quasi-uniform space*. If $\mathcal{U}$ is a quasi-uniformity on a set X, then $\mathcal{U}^{-1} = \{U^{-1} | U \in \mathcal{U}\}$ is also a quasi-uniformity on X called the *conjugate* of $\mathcal{U}$. Given a quasi-uniformity $\mathcal{U}$ on X, $\mathcal{U}^{\star} = \mathcal{U} \bigvee \mathcal{U}^{-1}$ will denote the coarsest uniformity on X which is finer than $\mathcal{U}$. If $U \in \mathcal{U}$, the entourage $U \cap U^{-1}$ of $\mathcal{U}^{\star}$ will be denoted by $U^{\star}$. Every quasi-uniformity $\mathcal{U}$ on X generates a topology $\tau(\mathcal{U})$. A neighborhood base for each point $x \in X$ is given by $\{U(x) | U \in \mathcal{U}\}$ where $U(x) = \{y \in X | (x,y) \in U\}$.

It is well known that there is a base for $\mathcal{U}$, which we always denote by $\mathcal{U}_0$, consisting of all $U \in \mathcal{U}$ that are $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-open in $X \times X$ (cf. [7, page 8]).

According to D.Doitchinov [5, Definition 1], a net $(y_\beta)_{\beta \in B}$ is called *a conet* of the net $(x_a)_{a \in A}$, if for any $U \in \mathcal{U}$ there are $a_U \in A$ and $\beta_U \in B$ such that $(y_\beta, x_a) \in U$ whenever $a \geq a_U$ and $\beta \geq \beta_U$. In this case we write $(y_\beta, x_a) \longrightarrow 0$ (see fig.1).

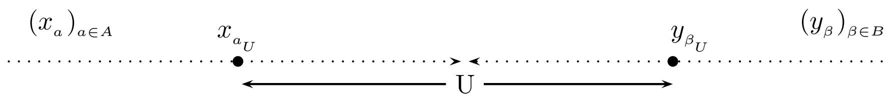


Figure 1

We preserve the Doitchinov's duality of net-conet and we will make use of a notion given by Kelly ([8, page 75]) under the terminology of p-sequence and q-sequence referring to the two directions of a net, and by Stoltenberg ([13, page 229]) who worked with only the one direction.

Definition 1. A net $(x_a)_{a \in A}$ in a quasi-uniform space $(X,\mathcal{U})$ is called τ_p*-net* (resp. τ_q*-net*) *for* W, if there is an $a_W \in A$ such that $(x_{a'}, x_a) \in W$ (resp. $(x_a, x_{a'}) \in W$) for each $a' \geq a \geq a_W$. The net $(x_a)_{a \in A}$ is called τ_p*-net* (resp. τ_q*-net*), if for each $W \in \mathcal{U}$, it is a τ_p*-net* (resp. τ_q*-net*) for W. We call x_{a_W} *extreme point for* W of the τ_p-net (resp. τ_q-net) and the a_W *extreme index for* V (see fig. 2).

We will make use of the phrase "*final segment of a* τ_p*-net*": we mean, the set of the elements of the τ_p-net from one point of it onwards (*until the end*). We dually define the "*final segment of a* τ_q*-net*".

We also say for a net $(x_a)_{a \in A}$, $t \in X$ and $U \in \mathcal{U}$:

"*finally* $(t, (x_a)_a) \in U$ "or, in symbols, "$\tau.(t, (x_a)_a) \in U$",

if $(t, x_a) \in U$ for all the points x_a of a final segment of $(x_a)_{a \in A}$.

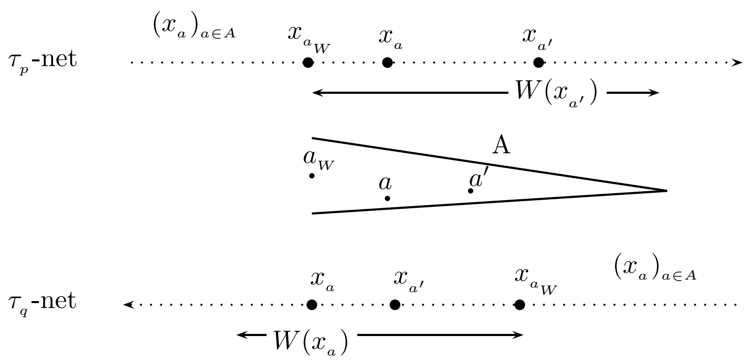


Figure 2

We give a similar meaning in the notations: $\tau.((y_\beta)_\beta, t) \in U$ and in the same way $\tau.((y_\beta)_\beta, (x_a)_a) \in U$ (see fig. 3).

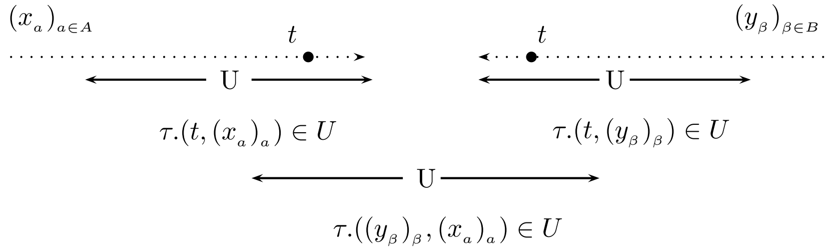


Figure 3

Remark 2. It is not obligatory for a convergent net to get subnets which are τ_p-nets or τ_q-nets.

Definition 3. Let $(\mathcal{A}, \mathcal{B})$ be an ordered couple whose members are non-empty families of τ_p- and τ_q-nets respectively. We say that $(\mathcal{A}, \mathcal{B})$ is a τ-*cut* (*of nets*) if the following conditions are fulfilled:

(1) for every $U \in \mathcal{U}$, every $(x^i_a)_{a\in A_i} \in \mathcal{A}$ and every $(y^j_\beta)_{\beta\in B_j} \in \mathcal{B}$ there holds

$$\tau.((y^j_\beta)_\beta, (x^i_a)_a) \in U.$$

(2) $\mathcal{B}$ contains all the τ_q-nets which are conets of all the τ_p-nets of $\mathcal{A}$ and conversely: $\mathcal{A}$ contains all the τ_p-nets whose conets are all the τ_q-nets elements of $\mathcal{B}$.

We call the member $\mathcal{A}$ (resp. $\mathcal{B}$) *first* (resp.*second) class of the τ-cut* and the elements of $\mathcal{A}$ and $\mathcal{B}$, *elements of the τ-cut.*

Throughout the paper, for simplicity of the proofs, we call the elements of $\mathcal{B}$ τ_q-*conets* of $\mathcal{A}$. Moreover, by saying that a τ_p-net (resp. τ_q-conet) member of the first (resp. second) class of a τ-cut converges to a point $x \in X$, we mean that it converges with respect to $\tau(\mathcal{U})$ (resp. $\tau(\mathcal{U}^{-1})$) (see fig. 4 and 6).

$(\mathcal{A}, \mathcal{B})$ τ-cut

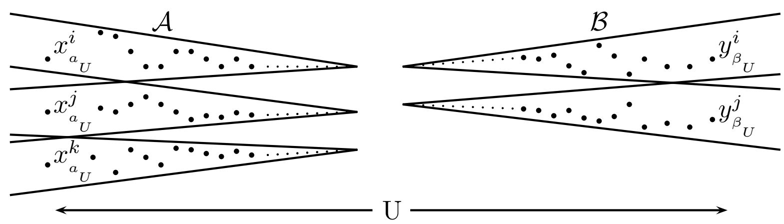


Figure 4

If all the elements of $\mathcal{A}$ and $\mathcal{B}$ converge to a point $x \in X$, then $\{x\}$ belongs to both of the classes $\mathcal{A}$ and $\mathcal{B}$. In this case, we call x *end of the two classes* or we say that *both of the classes have x as an end point*, or we simply say that x *is an end point of the τ-cut* $(\mathcal{A}, \mathcal{B})$. If there is an end point we say that $(\mathcal{A}, \mathcal{B})$ *converges* to x. In the case of a uniform space the two classes coincide.

If the classes of $(\mathcal{A}, \mathcal{B})$ have not an end point we say that $(\mathcal{A}, \mathcal{B})$ is a τ-*gap*. The set of all τ-gaps of X is symbolized by $\Lambda(X)$.

Notation 4. We symbolize the set of all τ-cuts of X by $\overline{X}$. For every $\xi \in \overline{X}$, we put $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi)$, $\mathcal{A}_\xi$, $\mathcal{B}_\xi$ being the two classes of the τ-cut ξ. We will preserve this notation and terminology throughout the paper (see fig. 5).

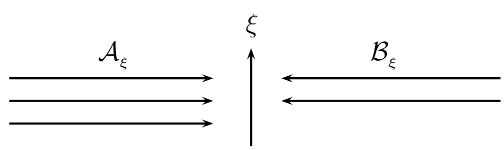


Figure 5

In a T_0 space X, for any two elements x and y, there is an entourage, say U, such that the one of the points, say y, does not belong to $U(x)$. It means that we may uniquely correspond to every point $x \in X$ a τ-cut $\phi(x) = (\mathcal{A}_{\phi(x)}, \mathcal{B}_{\phi(x)})$, where all the points of $\mathcal{A}_{\phi(x)}$ and $\mathcal{B}_{\phi(x)}$ converge to x. We recall that the point x itself, which is the "end"of all these τ_p-nets and τ_q-conets, belongs to both of the classes (see fig. 6).

We call the above map $\phi : X \longrightarrow \overline{X}$ "*the canonical embedding of X into $\overline{X}$*"and we preserve the notation and the meaning of that ϕ throughout all the paper.

So, from one point of view we may, roughly speaking, consider that $\phi(x)$ and $x \in X$ coincide, giving a reason why we write $X \subseteq \overline{X}$. By definition, $\overline{X} = \phi(X) \cup \Lambda(X)$.

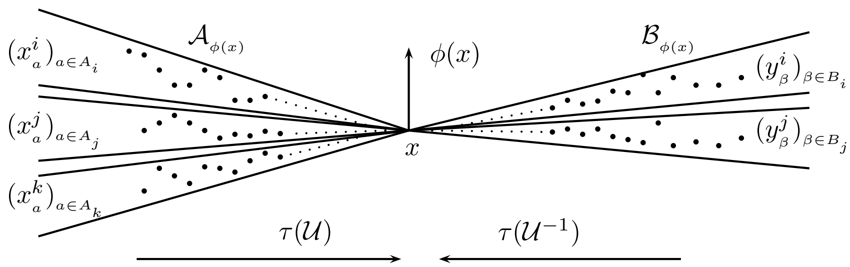


Figure 6

Remark 5. (i) There is a correspondence between a point of a T_0-space and a τ-cut (ii) If the τ_p-nets of the first class of a τ-cut converge to a point x and the τ_q-conets of the second class do not converge to x, then the τ-cut is not the $\phi(x)$. In this case the τ-cut is a τ-gap.

Consider the following example: in a τ-cut $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi)$, the class $\mathcal{A}_\xi$ consists of a τ_p-net A and the second class $\mathcal{B}_\xi$ consists of a τ_q-conet B and of a singleton $\{\beta\}$ as another τ_q-conet. Then, the singleton $\{\beta\}$ itself constitutes another τ-cut, the $\phi(\beta)$, different of the τ-cut $(\mathcal{A}_\xi, \mathcal{B}_\xi)$.

Proposition 6. For every τ-cut $(\mathcal{A}, \mathcal{B})$, every $U \in \mathcal{U}$ and every $(x^i_a)_{a\in A_i} \in \mathcal{A}$, $(y^j_\beta)_{\beta\in B_j} \in \mathcal{B}$, there are fixed indices $a^i_U \in A_i$, $\beta^j_U \in B_j$ such that $\tau.(y^j_\beta, x^i_a) \in U$ for every $a \geq a^i_U$ and $\beta \geq \beta^j_U$ (see fig. 7).

Proof. Let $V, U \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. Let also $(x^i_a)_{a\in A_i} \in \mathcal{A}$, $(y^j_\beta)_{\beta\in B_j} \in \mathcal{B}$ and a^i_V, β^j_V two indices such that $(x^i_a, x^i_{a'}) \in V$, $(y_\beta, y^j_{\beta'}) \in V$ for each $a, a' \geq a^i_V$ and $\beta, \beta' \geq \beta^j_V$. Then from $\tau.((y^j_\beta)_\beta, (x^i_a)_a) \in V$ we conclude that $(y^j_\beta, x^i_a) \in U$ for each $i \in I, j \in J, a \geq a^i_V$ and $\beta \geq \beta^j_V$. $\square$

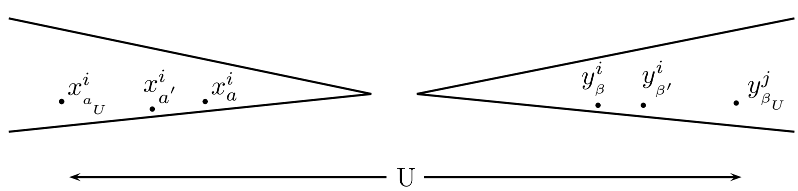


Figure 7

Remark 7. The existence in the Proposition 6 fixed extreme points for a given entourage U and for all pairs of the τ_p-nets and their τ_q-conets is a crucial property of the so called *quiet spaces*. It is a result coming from the $\mathcal{Q}$-property (the property of the quiet spaces) and the proposition 12 of [5] and it is the basic reason of why these spaces may be completed in such a simple way by the D-completion.

The following proposition is evident.

Proposition 8. A τ-cut is not a τ-gap if and only if all the members of the two classes converge to the same point.

Definition 9. We call τ-*Cauchy net* every τ_p-net or τ_q-conet of a τ-cut.

Definition 10. A quasi-uniform space is called τ-*complete* if all the τ-Cauchy nets of a τ-cut converge to the same point.

After the Proposition 8 an equivalent definition of the τ-completeness holds:

Definition 11. A quasi-uniform space is called τ-*complete* if its τ-cuts are not τ-gaps.

Definition 12. Let $(X,\mathcal{U})$ be a quasi-uniform space and let $(x_a)_{a\in A}$ and $(x_\beta)_{\beta\in B}$ be two nets in X. Given a $W\in\mathcal{U}$ we say that $(x_a)_{a\in A}$ and $(x_\beta)_{\beta\in B}$ are *left cofinal for* W if and only if there are $\beta_W\in B, a_W\in A$ satisfying the following property: for every $a\geq a_W$ there exists $\beta_a>\beta_W$ such that for every $\beta>\beta_a$ there holds $(x_\beta,x_a)\in W$ and for every $\beta\geq\beta_W$ there exists $a_\beta>a_W$ such that for every $a>a_\beta$ there holds $(x_a,x_\beta)\in W$.

We say that $(x_a)_{n\in A}$ and $(x_\beta)_{\beta\in B}$ are *left cofinal*, if there exists $W_0\in\mathcal{U}$ such that for each $W\in\mathcal{U}$ and $W\subseteq W_0$, $(x_a)_{a\in A}$ and $(x_\beta)_{\beta\in B}$ are left cofinal for W (see fig 8).

Dually we define the *right cofinality* of $(x_a)_{a\in A}$ and $(x_\beta)_{\beta\in B}$.

Proposition 13. Let $(x_a)_{a\in A}$ be a τ_p-net (resp.$(y_\beta)_{\beta\in B}$ is a τ_q-net) in a quasi-uniform space $(X,\mathcal{U})$ and $(x_{a_i})_{i\in I}$ a τ_p-subnet (resp. $(y_{\beta_j})_{j\in J}$ a τ_q-subnet) of it. Then $(x_a)_{a\in A}$ and $(x_{a_i})_{i\in I}$ (resp. $(y_\beta)_{\beta\in B}$ and $(y_{\beta_j})_{j\in J}$) are left (resp. right) cofinal.

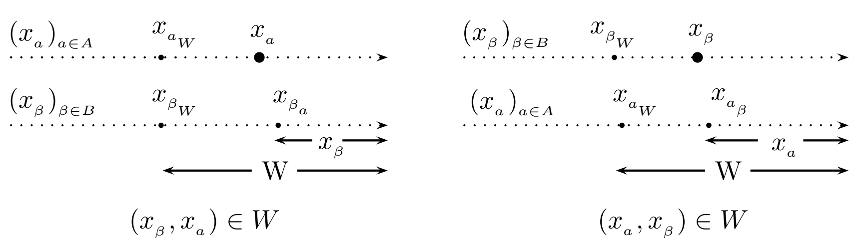


Figure 8

Proposition 14. In a quasi-uniform space two left cofinal τ_p-nets (resp. cofinal τ_q-conets) have the same τ_q-conets (resp. τ_p-nets).

Proof. Let $(x_a)_{a\in A}$, $(t_k)_{k\in K}$ be two left cofinal τ_p-nets and $(y_\beta)_{\beta\in B}$ is a τ_q-conet of $(x_a)_{a\in A}$. Let also $U\in\mathcal{U}$ and a $W\in\mathcal{U}$ with $W\circ W\subseteq U$. Then there exist $a_W\in A$ and $\beta_W\in B$ such that $(y_\beta,x_a)\in W$ for $a\geq a_W$ and $\beta\geq\beta_W$. On the other hand, because of left cofinality of $(x_a)_{a\in A}$ and $(t_k)_{k\in K}$, there are $a'_W\in A$ and $k_W\in K$ with the property: for each $k>k_W$ there exists $a_k>a'_W$ such that for each $a>a_k$ we have $(x_a,t_k)\in W$. Hence, $(y_\beta,t_k)\in W\circ W\subseteq U$, whenever $\beta\geq\beta_W$ and $k\geq k_W$.

Likewise we prove the result for the right cofinal τ_q-nets. □

Proposition 15. In a quasi-uniform space $(X,\mathcal{U})$ two left cofinal τ_p-nets (resp. right cofinal τ_q-conets) have the same limit points.

Remark 16.

(1) Without loss of generality, we may suppose that for $U\subseteq V$, it is $a^i_U\geq a^i_V$ and $\beta^j_U\geq\beta^i_V$ for the corresponded extreme points of $(x^i_a)_{a\in A_i}$ and $(y^j_\beta)_{\beta\in B_j}$ respectively.

(2) Given one or more τ_p-nets of a τ-cut we can construct the τ-cut taking all the τ_q-conets of the given and after that, all the τ_p-nets of these τ_q-conets. The procedure may be reversed considering firstly the τ_q-conets. In the present text we preserve the former procedure.

3. The τ-completion procedure

The pair $(X,\mathcal{U})$ always presents a T_0 quasi-uniform space, $\overline{X}$ is the set of all τ-cuts in X and ϕ the set-theoretical embedding of X in $\overline{X}$ described above in the notation 4 and the remark 5, the *canonical embedding of* X *into* $\overline{X}$ as we have called it. We shall define a quasi-uniformity on $\overline{X}$.

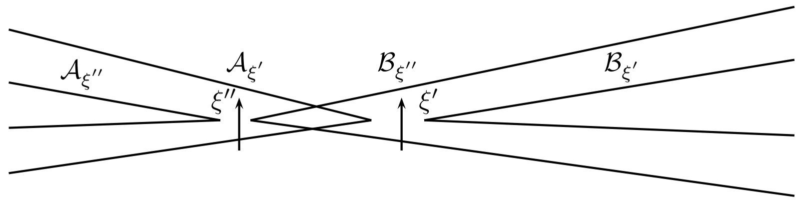
Definition 17. For any τ-cut $\xi'=(\mathcal{A}_{\xi'},\mathcal{B}_{\xi'})\in\overline{X}$ and any $W\in\mathcal{U}$ we define $\overline{W}(\xi')$ as the set of all $\xi''=(\mathcal{A}_{\xi''},\mathcal{B}_{\xi''})\in\overline{X}$ which fulfil the following:

(1) exclusively $\mathcal{A}_{\xi''}\subseteq\mathcal{A}_{\xi'}$ or

(2) there is a τ_p-net $(x_\beta)_{\beta\in B}\in A_{\xi'}$ such that $\tau.((x_\beta)_\beta,(x_a)_a)\in W$ for every $(x_a)_{a\in A}\in\mathcal{A}_{\xi''}$ (see fig. 9).

We put $\overline{\mathcal{U}}=\{\overline{W}:W\in\mathcal{U}\}$.

Let $\mathcal{U}_0$ be again the base of the quasi-uniformity the referred of the beginning of the paragraph 2. In the sequel, to every entourage in $(X,\mathcal{U})$, say $U\in\mathcal{U}$, the Definition 17 corresponds a subset of $\overline{X}\times\overline{X}$, that is an "entourage" in $(\overline{X},\overline{\mathcal{U}})$; we symbolize this subset by $\overline{U}$, that is by the same letter as in X putting a bar.



$\mathcal{A}_{\xi''} \subseteq \mathcal{A}_{\xi'}$

or

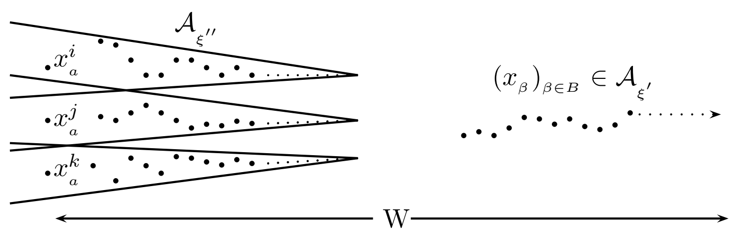


Figure 9

The following theorem ensures the existence of a topology in $\overline{X}$.

Theorem 18. The family $\overline{\mathcal{U}} = \{\overline{W} | W \in \mathcal{U}_0\}$ is a base for a quasi-uniformity on $\overline{X}$. We thus define a new quasi-uniform space $(\overline{X}, \overline{\mathcal{U}})$.

Proof. If U, V in $\mathcal{U}_0$ and $U \subseteq V$, then $\overline{U} \subseteq \overline{V}$. In fact; if $(\xi', \xi'') \in \overline{U}$ and $\mathcal{A}_{\xi''} \subseteq \mathcal{A}_{\xi'}$ the relation is evident. Otherwise, if $(\xi', \xi'') \in \overline{U}$, there is a τ_p-net $(x_\beta)_{\beta\in B}$ of ξ' such that $\tau.((x_\beta)_\beta, (x_a)_a) \in U$ for every $(x_a)_{a\in A} \in \mathcal{A}_{\xi''}$, hence $\tau.((x_\beta)_\beta, (x_a)_a) \in V$ and $(\xi', \xi'') \in \overline{V}$. From this result, we conclude that $\overline{U \cap V} \subseteq \overline{U} \cap \overline{V}$ and the considered family is a filter. We also have - by definition- that for every $\xi \in \overline{X}$ and every $\overline{W}$, $(\xi, \xi) \in \overline{W}$.

Let U, W be in $\mathcal{U}_0$ such that $W \circ W \subseteq U$, $(\xi', \xi) \in \overline{W}$ and $(\xi, \xi'') \in \overline{W}$. We will prove that $(\xi', \xi'') \in \overline{U}$. The only interesting case is if $\mathcal{A}_\xi \nsubseteq \mathcal{A}_{\xi'}$ and $\mathcal{A}_{\xi''} \nsubseteq \mathcal{A}_\xi$. Then, there is a τ_p-net $(x_a)_{a\in A} \in \mathcal{A}_{\xi'}$, whose the elements of a final segment fulfil $\tau.((x_a)_a, (z_\gamma)_\gamma) \in W$ for every $(z_\gamma)_{\gamma\in\Gamma} \in \mathcal{A}_\xi$. In a similar process, there is a final segment of a τ_p-net $(z_\gamma)_{\gamma\in\Gamma} \in \mathcal{A}_\xi$ such that $\tau.((z_\gamma)_\gamma, (y_\beta)_\beta) \in W$, for every $(x_\beta)_{\beta\in B} \in \mathcal{A}_{\xi''}$. In conclusion $\tau.((x_a)_a, (x_\beta)_\beta) \in W \circ W \subseteq U$ and this completes the proof. $\square$

Theorem 19. For each $U \in \mathcal{U}_0$, $(\phi(a), \phi(b)) \in \overline{U}$ if and only if $(a, b) \in U$.

Proof. Let $U \in \mathcal{U}_0$ and $(a,b) \in U$. Then, there is a $W \in \mathcal{U}_0$ such that $W^{-1}(a) \times W(b) \subseteq U$. Thus, for every τ_p-net $(x_\beta)_{\beta\in B}$ converging to b, we have $\tau.(a,(x_\beta)_\beta) \in U$ and since a is a τ_p-net of $\phi(a)$, we conclude that $(\phi(a),\phi(b)) \in \overline{U}$.

Conversely: let $(\phi(a),\phi(b)) \in \overline{U}$. Then there is a τ_p-net, say $(x_a)_{a\in A} \in \mathcal{A}_{\phi(a)}$, such that for every $(y^j_\beta)_{\beta\in B_j} \in \mathcal{A}_{\phi(b)}$ there holds $\tau.((x_a)_a,(y^j_\beta)_\beta) \in U$. Since b is a τ_p-net of $\phi(b)$ we have $\tau.((x_a)_a, b) \in U$. Since $U \in \mathcal{U}_0$ there is a $W \in \mathcal{U}_0$ such that $W^{-1}(x_a) \times W(b) \subseteq U$, for all the elements x_a of a final segment of $(x_a)_{a\in A}$. And since $(x_a)_{a\in A}$ converges to a, $a \in W^{-1}(x_a)$ and finally $(a,b) \in U$. $\square$

We also have the following:

Proposition 20. If $(x_a)_{a\in A}$ is a τ_p-net of a $\xi \in \overline{X}$, then $\lim_a \phi(x_a) = \xi$. Dually, if $(y_\beta)_{\beta\in B}$ is a τ_q-conet of a $\xi \in \overline{X}$, then $\lim_\beta(\phi(y_\beta),\xi) = 0$.

Proof. Let V, U in $\mathcal{U}_0$ such that $V \circ V \subseteq U$. We have that $(x_\gamma, x_a) \in V$, for each a,γ in A with $\gamma \geq a \geq a_V$ (a_V the extreme index of $(x_a)_{a\in A}$ for V). Fix an $a \geq a_V$ and pick a τ_p-net $(x^\kappa_\delta)_{\delta\in\Delta_\kappa}$ of $\phi(x_a)$. Then $x^\kappa_\delta \longrightarrow x_a$ and so $(x_a, x^\kappa_\delta) \in V$, whenever $\delta \geq \delta_0(\kappa)$ for some $\delta_0(\kappa) \in \Delta_\kappa$. Hence, $(x_\gamma, x^\kappa_\delta) \in U$ for $\gamma \geq a$ and $\delta \geq \delta_0(\kappa)$. Hence $(\xi,\phi(x_a)) \in \overline{U}$, whenever $a \geq a_V$.

The proof of the dual is similar. $\square$

Corollary 21. The set $\phi(X)$ is dense in $(\overline{X},\overline{\mathcal{U}})$ and $(\overline{X},(\overline{\mathcal{U}})^{-1})$.

Proof. The result comes directly from the above proposition, since, if $(x_a)_{a\in A} \in \mathcal{A}_\xi$ for a $\xi \in \overline{X} \setminus \phi(X)$, then $\lim_a \phi(x_a) = \xi$; thus, for any $U \in \mathcal{U}_0$, there are some $\phi(x_a)$ which belong to $\overline{U}(\xi)$. $\square$

Thus, the map $\phi : X \to \overline{X}$, apart of being a set theoretical embedding, is a topological embedding as well.

Definition 22. We call the structure $(\overline{X},\overline{\mathcal{U}})$, *the τ-complement of* $(X,\mathcal{U})$ and the whole process of this construction *the τ-completion of* $(X,\mathcal{U})$.

The more hard point of the τ-completion theory is the proof that the structure $(\overline{X},\overline{\mathcal{U}})$ is τ-complete. We follow the usual procedure: we consider a τ_p-net $(\xi_{a^\star})_{a^\star\in A^\star}$ (and one of its τ_q-conets $(\eta_{\beta^\star})_{\beta^\star\in B^\star}$) in $(\overline{X},\overline{\mathcal{U}})$ and we suppose that the τ_p-nets do not converge, which means that they have not any *end point.* So we define a suitable τ-cut in $(X,\mathcal{U})$, firstly a τ-cut depended upon an entourage (Lemma 23). Next we prove (Lemma 25) that, independently of this entourage a point of $\overline{X}$, say ξ, is fixed and that $(\xi_{a^\star})_{a^\star\in A^\star}$ converges to ξ. In the Lemmas 24 and 26 we refer to some τ_q-conets and, lastly, in theTheorem 28 we prove the main result: there is a τ-completion for any T_0 quasi-uniform space.

4. The basic lemmas for the construction of a τ-completion.

The quasi-uniformity $\mathcal{U}_0$ always is the referred in the beginning of the §2. For brevity and simplicity we make use of some phrases and some notation.

(1) If $(\xi_{a^\star})_{a^\star\in A^\star}$ in $\overline{X}$ is a τ_p-net in $\overline{X}$, then

$$(x_{\rho(\kappa_{a^\star})})_{\rho(\kappa_{a^\star})\in P_{(\kappa_{a^\star})}}$$

will denote a τ_p-net in $\mathcal{A}_{\xi_{a^\star}}$ for a concrete $a^\star \in A^\star$. More precisely, if $a^\star$ is a fixed index of $A^\star$, then $\kappa_{a^\star}$ denote the different τ_p-nets of $\xi_{a^\star}$ and $\rho(\kappa_{a^\star})$ denote the index set of each $\kappa_{a^\star}$ τ_p-net. Finally, $\rho_V(\kappa_{a^\star})$ denotes the extreme index of $(x_{\rho(\kappa_{a^\star})})_{\rho(\kappa_{a^\star})\in P_{(\kappa_{a^\star})}}$ for V (see fig. 10).

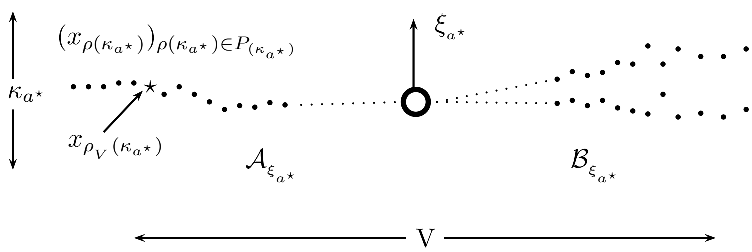


Figure 10

(2) Let I_{a_0} (resp. I_{β_0}) denote the final segment of $(x_a)_{a\in A}$ (resp. $(y_\beta)_{\beta\in B}$) with initial element x_{a_0} (resp. y_{β_0}). We say that the pair (I_{β_0}, I_{a_0}) is W*-close* if $(y_\beta, x_a) \in W$ whenever $\beta \geq \beta_0$ and $a \geq a_0$. By analogy we face the same problems for τ_q-conets.

We begin with a τ_p-net $(\xi_{a^\star})_{a^\star\in A^\star}$ in $\overline{X}$ and we advance in the construction of the demanded τ_p*-net for* W by transfinite induction on a well ordered set which, without loss of generality, may be considered as a subnet of $A^\star$.

Lemma 23. Let $(\xi_{a^\star})_{a^\star\in A^\star}$ be a τ_p-net in $(\overline{X},\overline{\mathcal{U}})$ without end point such that for each $\gamma^\star > a^\star$, $\mathcal{A}_{\xi_{a^\star}} \nsubseteq \mathcal{A}_{\xi_{\gamma^\star}}$. Let also $W \in \mathcal{U}$ be an entourage in $\mathcal{U}_0$. Then, there is a τ_p-net $(t_\lambda)_{\lambda\in\Lambda}$ such that the τ_p-nets $(\xi_{a^\star})_{a^\star\in A^\star}$ and $(\phi(t_\lambda))_{\lambda\in\Lambda}$ are left cofinal for $\overline{W}$.

Proof. Let $(\xi) = (\xi_{a^\star})_{a^\star\in A^\star}$ be as above, W, V be entourages of $\mathcal{U}_0$ such that $V \circ V \circ V \subseteq W$ and $\xi_{a^\star_{\overline{V}}}$ be the extreme point of (ξ) for $\overline{V}$. We symbolize by

$$(x_{\rho(\kappa_{a^\star})})_{\rho(\kappa_{a^\star})\in P_{\kappa_{a^\star}}}$$

any τ_p-net of $\xi_{a^\star}$, with the evident meaning of the notation and by $\rho_V(\kappa_{a^\star})$ the extreme index of $(x_{\rho(\kappa_{a^\star})})_{\rho(\kappa_{a^\star})\in P_{\kappa_{a^\star}}}$ for V. The entourage V is fixed during all the lemma's proof.

We advance to the construction of the demanded τ_p-*net for* W by transfinite induction on a well ordered set subnet of $A^\star$.

(1) *The first step.*

Let $\xi_{a_0^\star}$ in (ξ), $a_0^\star > a_V^\star$ and $(x_{\rho(\kappa^V_{a_0^\star})})_{\rho(\kappa^V_{a_0^\star})\in P_{\kappa^V_{a_0^\star}}}$ be an arbitrary τ_p-net of $\mathcal{A}_{\xi_{a_0^\star}}$.

Put $I_{\kappa^V_{a_0^\star}} = \{x_{\rho(\kappa^V_{a_0^\star})} | \rho(\kappa^V_{a_0^\star}) \geq \rho_V(\kappa^V_{a_0^\star})\}$ and $G_{a_0^\star} = I_{\kappa^V_{a_0^\star}}$.

Since (ξ) has not a last element, there are $a_1^\star > \gamma^\star > a_0^\star$ ($a_1^\star$ assigns to $\xi_{a_1^\star}$). Since $(\xi_{a_1^\star}, \xi_{\gamma^\star}) \in \overline{V}$ and $(\xi_{\gamma^\star}, \xi_{a_0^\star}) \in \overline{V}$ we have the following:

(A) There is a final segment for V

$$I_{\kappa^V_{a_1^\star}} = \{x_{\rho(\kappa^V_{a_1^\star})} | \rho(\kappa^V_{a_1^\star}) \geq \widetilde{\rho}(\kappa^V_{a_1^\star})\} \text{ with } \widetilde{\rho}(\kappa^V_{a_1^\star}) \geq \rho_V(\kappa^V_{a_1^\star})$$

of a τ_p-net $(x_{\rho(\kappa^V_{a_1^\star})})_{\rho(\kappa^V_{a_1^\star})\in P_{(\kappa^V_{a_1^\star})}} \in \mathcal{A}_{\xi_{a_1^\star}}$ for which there are final segments $I_{k_{\gamma^\star}}$ of all τ_p-nets of $\xi^\star_{\gamma^\star}$ such that the pair $(I_{\kappa^V_{a_1^\star}}, I_{k_{\gamma^\star}})$ is V-close.[1]

(B) There is a final segment for V

$$I_{k^0_{\gamma^\star}} = \{x_{\rho(k^0_{\gamma^\star})} | \rho(k^0_{\gamma^\star}) \geq \widetilde{\rho}(k^0_{\gamma^\star})\}$$

of a τ_p-net $(x_{\rho(k^0_{\gamma^\star})})_{\rho(k^0_{\gamma^\star})\in P_{(k^0_{\gamma^\star})}} \in \mathcal{A}_{\xi_{\gamma^\star}}$ for which there are final segments $\hat{I}_{k_{a_0^\star}}$ of all τ_p-nets of $\xi_{a_0^\star}$ such that the pair $(I_{k^0_{\gamma^\star}}, \hat{I}_{k_{a_0^\star}})$ is V-close.

(C) Since $(x_{\rho(\kappa^V_{a_0^\star})})_{\rho(\kappa^V_{a_0^\star})\in P_{(\kappa^V_{a_0^\star})}}$ is τ_p-net we have that

$$\hat{I}_{\kappa^V_{a_0^\star}} \subseteq I_{\kappa^V_{a_0^\star}} \text{ or } I_{\kappa^V_{a_0^\star}} \subseteq \hat{I}_{\kappa^V_{a_0^\star}}.$$

Finally, from (A) and (B) for $k_{\gamma^\star} = k^0_{\gamma^\star}$ we conclude that:

(D) a) The pair $(I_{\kappa^V_{a_1^\star}}, \hat{I}_{k_{a_0^\star}})$ is $V \circ V$-close for all $k_{a_0^\star}$, which jointly with (C) for $k_{a_0^\star} = \kappa^V_{a_0^\star}$ we have that:

b) The pair $(I_{\kappa^V_{a_1^\star}}, I_{\kappa^V_{a_0^\star}})$ is $V \circ V \circ V$-close (see fig.11).

We put $G_{a_1^\star} = G_{a_0^\star} \cup I_{\kappa^V_{a_1^\star}}$. Since (ξ) has not a last element, there are $a_2^\star, \gamma^{\star\star} \in A^\star$ such that $a_2^\star > \gamma^{\star\star} > a_1^\star > a_0^\star$.

[1]In the following, the entourage V in the form $\kappa^V_{a_i^\star}$ will indicate the concrete τ_p-net which have been chosen in the point $\xi^\star_{a_i^\star}$ satisfying (A).

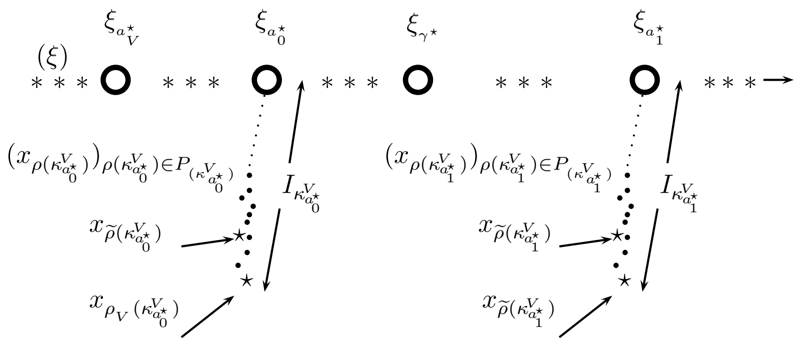


Figure 11

(E) From $a^\star_2 > \gamma^{\star\star} > a^\star_1$ and $a^\star_2 > \gamma^{\star\star} > a^\star_0$ according to the above process consisting of four steps ((A)→ (D)) we conclude that:
There is a final segment

$$I_{\kappa^V_{a^\star_2}} = \{x_{\rho(\kappa^V_{a^\star_2})} | \rho(\kappa^V_{a^\star_2}) \geq \widetilde{\rho}(\kappa^V_{a^\star_2})\} \text{ with } \widetilde{\rho}(\kappa^V_{a^\star_2}) \geq \rho_V(\kappa^V_{a^\star_2})$$

of a τ_p-net $(x_{\rho(\kappa^V_{a^\star_2})})_{\rho(\kappa^V_{a^\star_2})\in P_{(\kappa^V_{a^\star_2})}} \in \mathcal{A}_{\xi_{a^\star_2}}$, such that:

a) The pair $(I_{\kappa^V_{a^\star_i}}, \hat{I}_{k_{a^\star_j}})$ is $V \circ V$-close for all $k_{a^\star_j}$, $i, j \in \{0, 1, 2\}$, $i > j$ (for each $j \in \{0, 1\}$, $\hat{I}_{k_{a^\star_j}}$ are the final segments of the $k_{a^\star_j}$ τ_p-net of the point $\xi_{a^\star_j}$ which we take by applying the step (B) in the relations $(\xi_{\gamma^\star}, \xi_{a^\star_1}) \in \overline{V}$ and $(\xi_{\gamma^\star}, \xi_{a^\star_0}) \in \overline{V}$).

b) The pair $(I_{\kappa^V_{a^\star_i}}, I_{\kappa^V_{a^\star_j}})$ is $V \circ V \circ V$-close for each $i, j \in \{0, 1, 2\}$, $i > j$.

We put
(F) $G_{a^\star_2} = G_{a^\star_1} \cup I_{\kappa^V_{a^\star_2}}$ (see fig. 12).

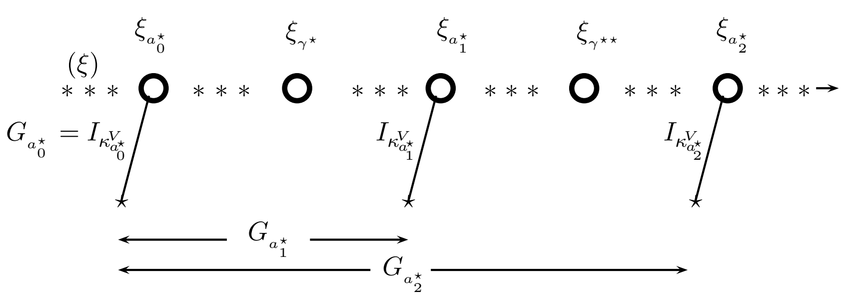


Figure 12

(2) *From* β *to* $\beta+1$.
We intend to pick up, by induction, a subnet $(\xi_{a_i^\star})_{i\in I_W}$ of (ξ) with the properties that have been generated in Step 1. We assume that β is a regular ordinal and that, for every $\xi_{a_i^\star}$, $i\le\beta$, we have already chosen: (i) The final segments $\hat{I}_{k_{a_i^\star}}$ of all the τ_p-nets of each $\xi_{a_i^\star}$ and (ii) the concrete final segment $I_{\kappa^V_{a_i^\star}}$ which has fixed extreme point, the point $x_{\widetilde{\rho}(\kappa^V_{a_i^\star})}$ $(\widetilde{\rho}(\kappa^V_{a_i^\star})\ge\rho_V(\kappa^V_{a_i^\star}))$. We then have:

(A_1) a) For each $j<i\le\beta$, the pair $(I_{\kappa^V_{a_i^\star}},\hat{I}_{k_{a_j^\star}})$ is $V\circ V$-close for all $k_{a_j^\star}$.

b) For each $j<i\le\beta$, the pair $(I_{\kappa^V_{a_i^\star}},I_{\kappa^V_{a_j^\star}})$ is $V\circ V\circ V$-close.

(B_1) Everyone of these points $\xi_{a_i^\star}$ corresponds to another set

$$G_{a_i^\star}=(\bigcup_{j<i}G_{a_j^\star})\cup I_{\kappa^V_{a_i^\star}}$$

Since $\xi_{a_\beta^\star}$ is not last element of (ξ), there are elements $\gamma^\star$ and $\epsilon^\star$ of $A^\star$such that $a_\beta^\star<\gamma^\star<\epsilon^\star$. Hence, for each $i\le\beta$ there holds $a_i^\star<\gamma^\star<\epsilon^\star$. But then, as in the above case (E), there is a concrete final segment

$$I_{k^V_{\epsilon^\star}}=\{x_{\rho(k^V_{\epsilon^\star})}|\rho(k^V_{\epsilon^\star})\ge\widetilde{\rho}(k^V_{\epsilon^\star})\}\text{ with }\widetilde{\rho}(k^V_{\epsilon^\star})\ge\rho_V(k^V_{\epsilon^\star})$$

of a τ_p-net $(x_{\rho(k^V_{\epsilon^\star})})_{\rho(k^V_{\epsilon^\star})\in P_{(k^V_{\epsilon^\star})}}\in\mathcal{A}_{\xi_{\epsilon^\star}}$ such that:

a) The pair $(I_{k^V_{\epsilon^\star}},\hat{I}_{k_{a_i^\star}})$ is $V\circ V$-close for each $i\le\beta$ and all $k_{a_i^\star}$.

b) The pair $(I_{k^V_{\epsilon^\star}},I_{\kappa^V_{a_i^\star}})$ is $V\circ V\circ V$-close for each $i\le\beta$.

We put $I_{\kappa^V_{a^\star_{\beta+1}}}=I_{k^V_{\epsilon^\star}}$ and $G_{a^\star_{\beta+1}}=(\bigcup_{i\le\beta}G_{a_i^\star})\cup I_{\kappa^V_{a^\star_{\beta+1}}}$.

It is evident that the above properties (A_1) to (B_1) are extended for each $i\le\beta+1$ (see fig. 13).

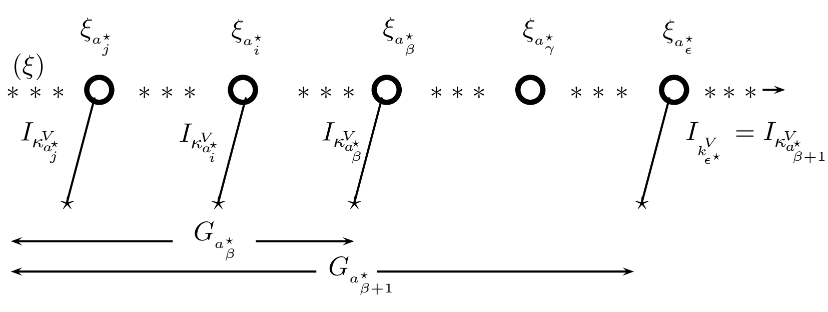


Figure 13

(3) *The case of being β a limit point.*

Let β be a limit ordinal and (ξ) be as above. We suppose that we have constructed a subnet $(\xi_{a_i^\star})_{i<\beta}$ of (ξ) whose the elements have indexes larger than $a^\star_{\overline{V}}$ and they constitute a linear subnet of $A^\star$. Moreover, for every one of these points, say $\xi_{a_i^\star}$, it corresponds the final segments $\hat{I}_{k_{a_i^\star}}$ of all the $k_{a_i^\star}$ τ_p-nets of $\xi_{a_i^\star}$ as well as the concrete final segment $I_{\kappa^V_{a_i^\star}}$ which has been chosen in the above process and it exclusively depends on V. That final segments have the following properties:

a$'$) The pair $(I_{\kappa^V_{a_i^\star}}, \hat{I}_{k_{a_j^\star}})$ is $V \circ V$-close for each $j < i < \beta$ and all $k_{a_j^\star}$.

b$'$) The pair $(I_{\kappa^V_{a_i^\star}}, I_{\kappa^V_{a_j^\star}})$ is $V \circ V \circ V$-close for each $j < i < \beta$.

If a cofinal linear subset of $A^\star$ has ordinal number the limit ordinal β, then the process is over. If it is not the case, there are elements $\gamma^\star$ and $\delta^\star$ of $A^\star$such that $a^\star_\beta < \gamma^\star < \delta^\star$. Hence $a^\star_i < \gamma^\star < \delta^\star$ for each $i < \beta$. But then, as in the above case (E), there is a final segment

$$I_{k^V_{\delta^\star}} = \{x_{\rho(k^V_{\delta^\star})} | \rho(k^V_{\delta^\star}) \geq \widetilde{\rho}(k^V_{\delta^\star})\} \text{ with } \widetilde{\rho}(k^V_{\delta^\star}) \geq \rho_V(k^V_{\delta^\star})$$

of a τ_p-net $(x_{\rho(k^V_{\delta^\star})})_{\rho(k^V_{\delta^\star}) \in P_{(k^V_{\delta^\star})}} \in \mathcal{A}_{\xi_{\delta^\star}}$ such that:

a$''$) The pair $(I_{k^V_{\delta^\star}}, \hat{I}_{k_{a_i^\star}})$ is $V \circ V$-close for each $i < \beta$ and all $k_{a_i^\star}$.

b$''$) The pair $(I_{k^V_{\delta^\star}}, I_{\kappa^V_{a_i^\star}})$ is $V \circ V \circ V$-close for each $i < \beta$.

We put $I_{\kappa^V_{a_\beta^\star}} = I_{k^V_{\delta^\star}}$ and $G^W_{a_\beta^\star} = (\bigcup\limits_{i<\beta} G^W_{a_i^\star}) \cup I_{\kappa^V_{a_\beta^\star}}$.

It is evident that the above properties (A_1) to (B_1) are extended for each $i < \beta$.

(4) *The continuation of the process.*

We continue the process until the end of (ξ), that is until the "exhausting"of the elements of (ξ) and we form the set $G^W = \cup\{G^W_\beta | \beta \text{ an ordinal}\}$. Thus, we have extracted from (ξ) a subnet $(\xi_{a_i})_{i \in I_W} = (\xi^W)$.

We consider the set

$$\widetilde{A}^V = \{\widetilde{a}^V \mid \widetilde{a}^V = \rho(\kappa^V_{a_i^\star}) \in P(\kappa^V_{a_i^\star}),\ i \in I_W\}$$

to which we give the following order:

$$\widetilde{a}^V = \rho(\kappa^V_{a_j^\star}) \leq \widetilde{a'}^V = \rho(\kappa^V_{a_i^\star})$$

if and only if

(i) $a^\star_j < a^\star_i$ or

(ii) $\kappa^V_{a^\star_j} = \kappa^V_{a^\star_i}$ and $\rho(\kappa^V_{a^\star_i}) = \rho'(\kappa^V_{a^\star_j}) \geq \rho(\kappa^V_{a^\star_j}) \geq \widetilde{\rho}(\kappa^V_{a^\star_j}) = \widetilde{\rho}(\kappa^V_{a^\star_i})$ (see fig. 14).

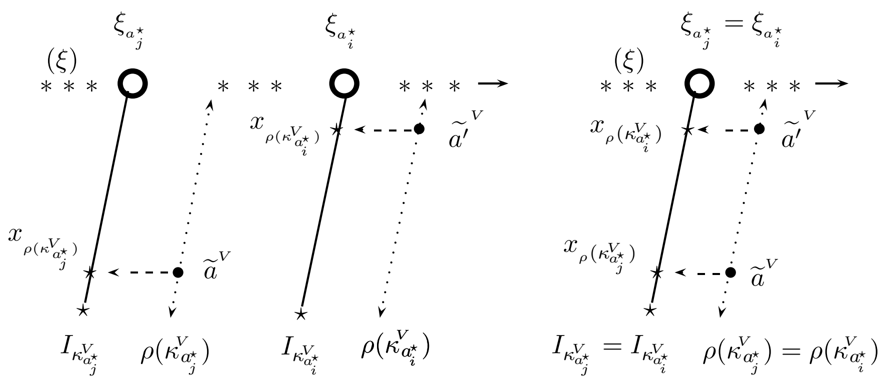


Figure 14

Thus, we have constructed the net $G^W = (x_{\rho(\kappa^V_{a^\star_i})})_{\rho(\kappa^V_{a^\star_i}) \in \widetilde{A}^V}$. It is easy to show that the validity of the Property (A_1) for each ordinal β imply that $(x_{\rho(\kappa^V_{a^\star_i})})_{\rho(\kappa^V_{a^\star_i}) \in \widetilde{A}}$ is a τ_p-net for W as well as $(\phi(x_{\rho(\kappa^V_{a^\star_i})}))_{\rho(\kappa^V_{a^\star_i}) \in \widetilde{A}^V}$ and $(\xi_{a^\star})_{a^\star \in A^\star}$ are left cofinal for $\overline{W}$ (see fig. 15). $\square$

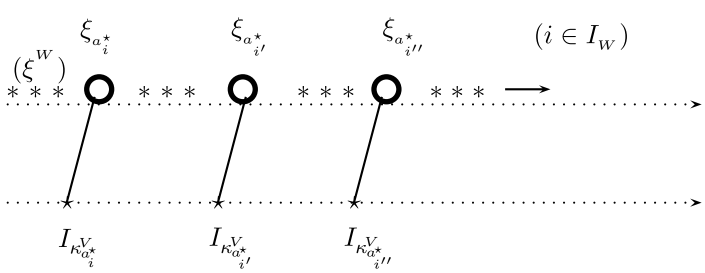


Figure 15

A similar demonstration gives the following:

Lemma 24. Let $(\eta_{\beta^\star})_{\beta^\star \in B^\star}$ be a τ_q-conet in $(\overline{X}, \overline{\mathcal{U}})$ without end point such that for each $\delta^\star > \beta^\star$, $\mathcal{A}_{\eta_{\delta^\star}} \not\subseteq \mathcal{A}_{\eta_{\beta^\star}}$. Let also $W \in \mathcal{U}$ be an entourage in $\mathcal{U}_0$. Then, there is a τ_q-conet for W, $(s_\mu)_{\mu \in M}$, such that the nets $(\eta_{\beta^\star})_{\beta^\star \in B^\star}$ and $(\phi(s_\mu))_{\mu \in M}$ are right cofinal for W.

Lemma 25. Let $(\xi_{a^\star})_{a^\star\in A^\star}$ be a τ_p-net in $(\overline{X},\overline{\mathcal{U}})$ without end point such that for each $\gamma^\star > a^\star$, $\mathcal{A}_{\xi_{a^\star}} \nsubseteq \mathcal{A}_{\xi_{\gamma^\star}}$. Then, there is a τ_p-net $(t_\lambda)_{\lambda\in\Lambda}$ of $(X,\mathcal{U})$ such that the nets $(\xi_{a^\star})_{a^\star\in A^\star}$ and $(\phi(t_\lambda))_{\lambda\in\Lambda}$ are left cofinal.

Proof. Let $(\xi) = (\xi_{a^\star})_{a^\star\in A^\star}$ be as above. By the construction of Lemma 23, for each $W \in \mathcal{U}$, we consider: (1) The subnet $(\xi^W) = (\xi_{a^\star_i})_{i\in I_W} = (\xi_{a^\star_{i(W)}})_{i(W)\in I_W}$ of (ξ); (2) The net $G^W = (x_{\rho(\kappa^V_{a^\star_{i(W)}})})_{\rho(\kappa^V_{a^\star_{i(W)}})\in \tilde{A}^V}$ of X which corresponds to a $W \in \mathcal{U}_0$ $(V \in \mathcal{U}_0,\ V\circ V\circ V \subseteq W)$;[2] (3) The final segments $I_{\kappa^V_{a^\star_{i(W)}}}$ as well as their extreme points $x_{\widetilde{\rho}(\kappa^V_{a^\star_{i(W)}})}$ $(\widetilde{\rho}(\kappa^V_{a^\star_{i(W)}}) \geq \rho_V(\kappa^V_{a^\star_{i(W)}}))$.

(1) *The construction of the desired τ_p-net G.*

We put $G = \bigcup\{G^W \mid W \in \mathcal{U}\}$.

We consider the set

$$\overline{A} = \{\overline{a} = \rho(\kappa^V_{a^\star_{i(W)}}) | \rho(\kappa^V_{a^\star_{i(W)}}) \geq \widetilde{\rho}(\kappa^V_{a^\star_{i(W)}}), W \in \mathcal{U}\}$$

to which we give the following order:

$$\overline{a} = \rho_0(\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}) \leq \overline{a'} = \rho_1(\kappa^{V_\pi}_{a^\star_{i(W_\pi)}})$$

if and only if

(i) $a^\star_{i(W_\sigma)} < a^\star_{i(W_\pi)}$ and there exists a point $\xi_{\gamma^\star} = \xi_{a^\star_{i_{\gamma^\star}(W_\pi)}} \in (\xi_{a^\star_{i(W_\pi)}})_{i(W_\pi)\in I_{W_\pi}}$ such that $a^\star_{i(W_\sigma)} \leq a^\star_{i_{\gamma^\star}(W_\pi)} < a^\star_{i(W_\pi)}$, (it is assumed that if $W_\pi \subseteq W_\sigma$, then $V_\pi \subseteq V_\sigma$ as well) (see fig. 16)

or

(ii) $a^\star_{i(W_\pi)} = a^\star_{i(W_\sigma)}$, $\kappa^{V_\pi}_{a^\star_{i(W_\pi)}} = \kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}$ and $\overline{a'} = \rho_1(\kappa^{V_\pi}_{a^\star_{i(W_\pi)}}) = \rho_2(\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}) \geq \rho_0(\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}})$ (see fig. 17).

It is clear that $\overline{A}$ is nonempty. The relation $\leq$ is a right filtering preorder and hence the index-set $\overline{A}$ is directed. We only prove the transitivity of $\leq$, the rest are trivial. Indeed, let $\overline{c} = \rho^*(\kappa^{V_\theta}_{a^\star_{i(W_\theta)}})$ be such that $\overline{a} \leq \overline{a'}$ and $\overline{a'} \leq \overline{c}$. The only interesting case is if $a^\star_{i(W_\sigma)} < a^\star_{i(W_\pi)} < a^\star_{i(W_\theta)}$ with $V_\theta \subseteq V_\pi \subseteq V_\sigma$. Then, there are $\xi^\star_{\gamma^\star} \in (\xi_{a^\star_{i(W_\pi)}})_{i(W_\pi)\in I_{W_\pi}}$ and $\xi^\star_{\delta^\star} \in (\xi_{a^\star_{i(W_\theta)}})_{i(W_\theta)\in I_{W_\theta}}$ such that $a^\star_{i(W_\sigma)} \leq \gamma^\star < a^\star_{i(W_\pi)} \leq \delta^\star < a^\star_{i(W_\theta)}$. Because of the existence of that $\delta^\star$, we have $\overline{a'} \leq \overline{c}$.

[2] We remind that the index V in the form $\kappa^V_{a^\star_{i(W)}}$ refers to the different τ_p-nets which have been chosen in the concrete points $\xi_{a^\star}$ in the process of Lemma 23 with W (and hence V) changeable. If $W' \neq W$, then $a^\star_{i(W')} = a^\star_{i(W)} = a^\star$ does not imply $\kappa^{V'}_{a^\star_{i(W')}} = \kappa^V_{a^\star_{i(W)}}$ in general.

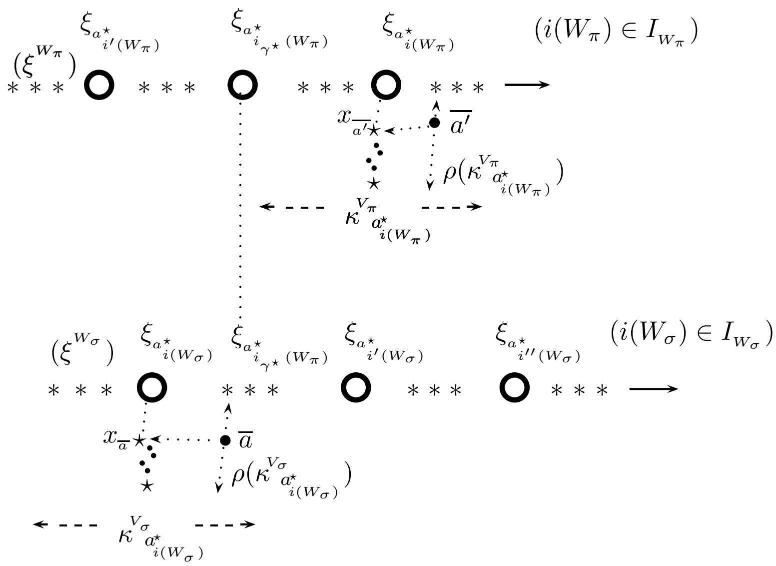


Figure 16

(2) *The proof that G is a τ_p-net.*

Let $W \in \mathcal{U}_0$. Suppose that $V = W_1 \in \mathcal{U}_0$ is the corresponded entourage to W from the lemma 23. Similarly we consider the entourage V_1 which corresponds to W_1. There holds $V \circ V \circ V \subseteq W$ and $V_1 \circ V_1 \circ V_1 \subseteq W_1$. Let $a^\star_{\overline{V}_1}$ be the extreme index of (ξ) for $\overline{V}_1$. Let $a^\star_\lambda \geq a^\star_{\overline{V}_1}$ with $\xi_{a^\star_\lambda} = \xi_{a^\star_{i_0(W_1)}} \in (\xi_{a^\star_{i(W_1)}})_{i(W_1) \in I_{W_1}}$ and let $\overline{a'} = \rho_1(\kappa^{V_\pi}_{a^\star_{i(W_\pi)}}) \geq \overline{a} = \rho_0(\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}) \geq \overline{a}_0 = \widetilde{\rho}(\kappa^{V_1}_{a^\star_{i_0(W_1)}})$ (there holds $a^\star_{\overline{V}} \leq a^\star_{i_0(W_1)} \leq a^\star_{i(W_\sigma)} \leq a^\star_{i(W_\pi)}$ and $W_\pi \subseteq W_\sigma \subseteq W_1 \subseteq W$). We distinguish two cases: (i) $a^\star_{i(W_\pi)} = a^\star_{i(W_\sigma)}$, $\kappa^{V_\pi}_{a^\star_{i(W_\pi)}} = \kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}$ and $\rho_1(\kappa^{V_\pi}_{a^\star_{i(W_\pi)}}) = \rho_2(\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}) \geq \rho_0(\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}) \geq \widetilde{\rho}(\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}})$. Then

$$(x_{\rho_1(\kappa^{V_\pi}_{a^\star_{i(W_\pi)}})}, x_{\rho_0(\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}})}) \in V_\sigma \subseteq W.$$

(ii) $a^\star_{i(W_\sigma)} < a^\star_{i(W_\pi)}$, $V_\pi \subseteq V_\sigma$ and there exists a point $\xi^\star_{\gamma^\star} = \xi_{a^\star_{i_{\gamma^\star}(W_\pi)}} \in (\xi_{a^\star_{i(W_\pi)}})_{i(W_\pi) \in I_{W_\pi}}$ such that $a^\star_{i(W_\sigma)} \leq a^\star_{i_{\gamma^\star}(W_\pi)} < a^\star_{i(W_\pi)}$. Then, from the extended (B_1) a)-property (for each ordinal β) in the Lemma 23, we conclude that

$$(x_{\rho(\kappa^{V_\pi}_{a^\star_{i(W_\pi)}})}, t) \in V_\pi \circ V_\pi \qquad (1)$$

for each $t \in \hat{I}_{k_{a^\star_{i_{\gamma^\star}(W_\pi)}}}$ ($\hat{I}_{k_{a^\star_{i_{\gamma^\star}(W_\pi)}}}$ are final segments of the $k_{a^\star_{i_{\gamma^\star}(W_\pi)}}$ τ_p-nets of $\xi_{a^\star_{i_{\gamma^\star}(W_\pi)}}$). Since $(\xi_{a^\star_{i_{\gamma^\star}(W_\pi)}}, \xi_{a^\star_{i(W_\sigma)}}) \in \overline{W}_\sigma$ there is a final segment I of a τ_p-net of $\xi_{a^\star_{i_{\gamma^\star}(W_\pi)}}$, $I \subset \hat{I}_{k_{a^\star_{i_{\gamma^\star}(W_\pi)}}}$ for some $k_{a^\star_{i_{\gamma^\star}(W_\pi)}} \in I_{W_\pi}$, for which there are final segments $I^\star_{k_{a^\star_{i(W_\sigma)}}}$ of all τ_p-nets of $\xi_{a^\star_{i(W_\sigma)}}$ such that the pair $(I, I^\star_{k_{a^\star_{i(W_\sigma)}}})$ is finally V_σ-close. Hence,

$$(I, I^\star_{\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}}) \text{ is finally } V_\sigma\text{-close} \quad (2).$$

Finally, we have

$$I^\star_{\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}} \subseteq I_{\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}} \text{ or } I_{\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}} \subseteq I^\star_{\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}}} \qquad (3).$$

From (1), (2) and (3) we conclude that

$$(x_{\rho(\kappa^{V_\pi}_{a^\star_{i(W_\pi)}})}, x_{\rho(\kappa^{V_\sigma}_{a^\star_{i(W_\sigma)}})}) \in V_\pi \circ V_\pi \circ V_\sigma \circ V_\sigma \subseteq W.$$

The index $\overline{a}_0 = \widetilde{\rho}(\kappa^{V_1}_{a^\star_{i_0(W_1)}})$ is the extreme index of G for W. Thus G is a τ_p-net in X. $\square$

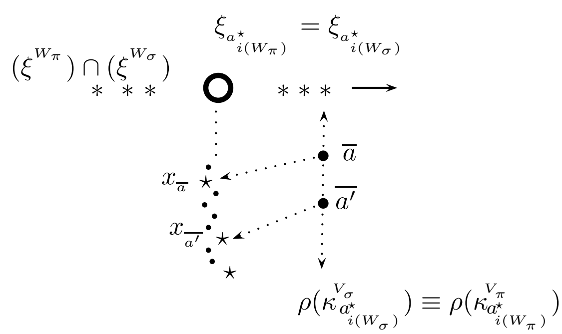


Figure 17

Lemma 26. Let $(\eta_{\beta^\star})_{\beta^\star \in B^\star}$ be a τ_q-conet in $(\overline{X}, \overline{\mathcal{U}})$ without end point such that for each $\delta^\star > \beta^\star$, $\mathcal{A}_{\eta_{\delta^\star}} \nsubseteq \mathcal{A}_{\eta_{\beta^\star}}$. Then, there is a τ_q-conet $(s_\mu)_{\mu \in M}$ of $(X, \mathcal{U})$ such that the nets $(\eta_{\beta^\star})_{\beta^\star \in B^\star}$ and $(\phi(s_\mu))_{\mu \in M}$ are right cofinal.

We state now the general case.

Theorem 27. For every τ-cut $\xi^\star$ in $(\overline{X},\overline{\mathcal{U}})$, there is a τ-cut ξ in $(X,\mathcal{U})$ such that each τ_p-net of $\mathcal{A}_{\xi^\star}$ $\tau((\overline{\mathcal{U}}))$-converges to ξ and each τ_q-conet of $\mathcal{B}_{\xi^\star}$ $\tau(\overline{\mathcal{U}^{-1}})$-converges to ξ.

Proof. Let $\xi^\star = (\mathcal{A}_{\xi^\star}, \mathcal{B}_{\xi^\star})$ be a τ-cut in $(\overline{X},\overline{\mathcal{U}})$ such that $\mathcal{A}_{\xi^\star} = \{(\xi^i_k)_{k\in K_i} | i \in I\}$ and $\mathcal{B}_{\xi^\star} = \{(\eta^j_\lambda)_{\lambda\in\Lambda_j} | j \in J\}$. We define

$\mathcal{A}_\xi = \{(x_a)_{a\in A} \,|(x_a)_{a\in A}$ τ_p-net in X and $(\phi(x_a)_{a\in A} \in \mathcal{A}_{\xi^\star}\}$ and

$\mathcal{B}_\xi = \{(y_\beta)_{\beta\in B} \,|(y_\beta)_{\beta\in B}$ τ_q-conet in X and $(\phi(y_\beta)_{\beta\in B} \in \mathcal{B}_{\xi^\star}\}$ (see fig. 18).

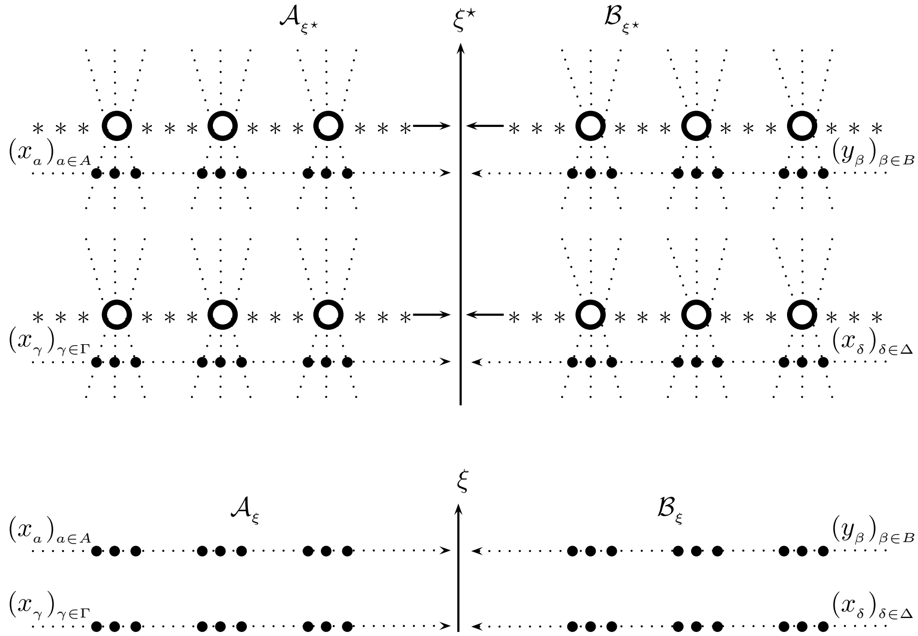


Figure 18

We first verify that the pair $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi)$ constitutes a τ-cut in $(X, \mathcal{U})$. It is need to be proved:
(A) *The classes* $\mathcal{A}_\xi$, $\mathcal{B}_\xi$ *are non-void* and
(B) *The pair* $(\mathcal{A}_\xi, \mathcal{B}_\xi)$ *satisfies the two conditions of the Definition 3.*

For (A): we consider $(\xi^i_k)_{k\in K_i} \in \mathcal{A}_{\xi^\star}$. We distinguish two cases.

(a). *The* τ_p*-net* $(\xi^i_k)_{k\in K_i}$ *is finally constant.*

In this case, there exists $k_0 \in K_i$ such that $\xi^i_k = \xi^i_{k_0}$ for every $k \geq k_0$. Suppose that $(x_a)_{a\in A} \in \mathcal{A}_{\xi^i_{k_0}}$. Then, Proposition 20 implies that $(\xi^i_{k_0}, (\phi(x_a))_a) \longrightarrow 0$ which jointly to $(\eta^j_\lambda, \xi^i_{k_0}) \longrightarrow 0$ we conclude that $(\eta^j_\lambda, (\phi(x_a))_a) \longrightarrow 0$. Thus $(x_a)_{a\in A} \in \mathcal{A}_{\xi^\star}$.

(b). *The* τ_p*-net* $(\xi^i_k)_{k\in K_i}$ *is not finally constant.*
We distinguish two subcases.
(b_1) *There is a* $k_0 \in K_i$ *such that for each* $k' > k \geq k_0$, $\mathcal{A}_{\xi^i_k} \subseteq \mathcal{A}_{\xi^i_{k'}}$.
Suppose that $(x_a)_{a\in A} \in \mathcal{A}_{\xi^i_{k_0}}$. Then, from $((\eta^j_\lambda)_\lambda, (\xi^i_k)_k) \longrightarrow 0$, $((\xi^i_k)_k, \xi^i_{k_0}) \longrightarrow 0$ and $(\xi^i_{k_0}, (\phi(x_a))_a) \longrightarrow 0$ we conclude that $((\eta^j_\lambda)_\lambda, (\phi(x_a))_a) \longrightarrow 0$ and thus $(x_a)_{a\in A} \in \mathcal{A}_\xi$.

(b_2) *For each* $k \in K$ *there is a* $k' > k$ *such that* $\mathcal{A}_{\xi^i_k} \not\subseteq \mathcal{A}_{\xi^i_{k'}}$.
In this case we can find a τ-subnet $(\xi^i_{k_\mu})_{\mu\in M}$ of $(\xi^i_k)_{k\in K_i}$ such that for each $\mu' > \mu$, it is $\mathcal{A}_{\xi^i_{k_\mu}} \not\subseteq \mathcal{A}_{\xi^i_{k_{\mu'}}}$. According to Lemma 23, there exists a τ_p-net $(x_a)_{a\in A}$ in $(X, \mathcal{U})$ whose the ϕ-images constitutes a τ_p-net left cofinal to $(\xi^i_{k_\mu})_{\mu\in M}$ and finally left cofinal to $(\xi^i_k)_{k\in K_i}$ (Proposition 13). Since $(\xi^i_k)_{k\in K_i} \in \mathcal{A}_{\xi^\star}$, Proposition 14 implies that $(\phi(x_a))_{a\in A} \in \mathcal{A}_{\xi^\star}$. Hence, $(x_a)_{a\in A} \in \mathcal{A}_\xi$. Thus $\mathcal{A}_\xi$ is non-void. Similarly it is proved that $\mathcal{B}_\xi$ is non-void.

For (B): we firstly prove that the members of $\mathcal{A}_\xi$ has as τ_q-conets all the members of $\mathcal{B}_\xi$. Indeed, by the construction of $\mathcal{A}_\xi$ and $\mathcal{B}_\xi$ we have that $((\phi(y_\beta))_\beta, (\phi(x_a))_a) \longrightarrow 0$. But then, Theorem 19 implies that $((y_\beta)_\beta, (x_a)_a) \longrightarrow 0$. So, the first demand for the being ξ a τ-cut is fulfilled.

For the second demand, let us assume that $(t_\gamma)_{\gamma\in\Gamma}$ is a τ_p-net which has as τ_q-conets all the members of $\mathcal{B}_\xi$. We have to prove that $(t_\gamma)_{\gamma\in\Gamma} \in \mathcal{A}_\xi$, that is, $((\eta^j_\lambda)_\lambda, (\phi(t_\gamma))_\gamma) \longrightarrow 0$ for each $j \in J$. Fix a $j \in J$. We distinguish two cases.

(a′). *The* τ_q*-conet* $(\eta^j_\lambda)_{\lambda\in\Lambda_j} \in \mathcal{B}_{\xi^\star}$ *is constant.*
Suppose that $\eta^j_\lambda = \eta^j_{\lambda_0}$ for each $\lambda \geq \lambda_0$. If $(\xi^i_k)_{k\in K_i} \in \mathcal{A}_{\xi^\star}$, then $(\eta^j_{\lambda_0}, (\xi^i_k)_k) \longrightarrow 0$. On the other hand, if $(y_\beta)_{\beta\in B} \in \mathcal{B}_{\eta^j_{\lambda_0}}$, then $((\phi(y_\beta))_\beta, \eta^j_{\lambda_0}) \longrightarrow 0$ and by

$(\xi^i_k)_{k\in K} \in \mathcal{A}_{\xi^\star}$ we conclude that $((\phi(y_\beta))_\beta, (\xi^i_k)_k) \longrightarrow 0$. Therefore, $(y_\beta)_{\beta\in B} \in \mathcal{B}_\xi$ which implies that $\mathcal{B}_{\eta^j_{\lambda_0}} \subseteq \mathcal{B}_\xi$. But then, $(t_\gamma)_{\gamma\in\Gamma}$ has as τ_q-conets all the members of $\mathcal{B}_{\eta^j_{\lambda_0}}$ which concludes that $(t_\gamma)_{\gamma\in\Gamma} \in \mathcal{A}_{\eta^j_{\lambda_0}} = \mathcal{A}_{\eta^j_\lambda}$. The last conclusion implies that that $((\eta^j_\lambda)_\lambda, (\phi(t_\gamma))_\gamma) \longrightarrow 0$, as it is desired.

(b′). *The τ_q-conet $(\eta^j_\lambda)_{\lambda\in\Lambda_j}$ is non-constant.*
We distinguish two subcases.

(b′$_1$). *There is a $\lambda_0 \in \Lambda$ such that for each $\lambda' > \lambda \geq \lambda_0$, $\mathcal{A}_{\eta^j_{\lambda'}} \subseteq \mathcal{A}_{\eta^j_\lambda}$.*
In this case, we have $\mathcal{B}_{\eta^j_\lambda} \subseteq \mathcal{B}_{\eta^j_{\lambda'}}$ and for a fixed $\lambda \geq \lambda_0$ and a $(y_\beta)_{\beta\in B} \in \mathcal{B}_{\eta^j_\lambda}$ we have $(y_\beta)_{\beta\in B} \in \mathcal{B}_{\eta^j_{\lambda'}}$ for each $\lambda' \geq \lambda$. From $((\phi(y_\beta))_\beta, (\eta^j_{\lambda'})_{\lambda'}) \longrightarrow 0$ and $((\eta^j_{\lambda'})_{\lambda'}, (\xi^i_k)_k) \longrightarrow 0$, we conclude that $((\phi(y_\beta))_\beta, (\xi^i_k)_k) \longrightarrow 0$. Hence, $(y_\beta)_{\beta\in B} \in \mathcal{B}_{\xi^\star}$. As in the proof for the case (a′), we conclude that $((\eta^j_\lambda)_\lambda, (\phi(t_\gamma))_\gamma) \longrightarrow 0$.

(b′$_2$). *For each $\lambda \in \Lambda$ there is $\lambda' > \lambda$ such that $\mathcal{A}_{\eta^j_{\lambda'}} \nsubseteq \mathcal{A}_{\eta^j_\lambda}$.*
In this case, for each $j \in J$, we can find a τ-subnet $(\eta^j_{\lambda_\sigma})_{\sigma\in\Sigma}$ such that for every $\sigma' > \sigma$ there holds $\mathcal{A}_{\eta^j_{\lambda_{\sigma'}}} \nsubseteq \mathcal{A}_{\eta^j_{\lambda_\sigma}}$. Then, from the Lemma 26, there exists a τ_q-conet $(y_\beta)_{\beta\in B}$ in $(X,\mathcal{U})$ such that the τ_q-conets $(\eta^j_{\lambda_\sigma})_{\sigma\in\Sigma}$ and $(\phi(y_\beta))_{\beta\in B}$ are right cofinal. Since $(\eta^j_\lambda)_{\lambda\in\Lambda} \in \mathcal{B}_{\xi^\star}$, Proposition 14 implies that $(\phi(y_\beta))_{\beta\in B} \in \mathcal{B}_{\xi^\star}$. Hence, $(y_\beta)_{\beta\in B} \in \mathcal{B}_\xi$. Therefore, $((y_\beta)_\beta, (t_\gamma)_\gamma) \longrightarrow 0$ or equivalently $((\phi(y_\beta))_\beta, (\phi(t_\gamma))_\gamma) \longrightarrow 0$. But then, since $(\eta^j_{\lambda_\sigma})_{\sigma\in\Sigma}$ and $(\phi(y_\beta))_{\beta\in B}$ are right cofinal we conclude that $((\eta^j_\lambda)_\lambda, (\phi(t_\gamma))_\gamma) \longrightarrow 0$. Hence, in any case we have that $((\eta^j_\lambda)_\lambda, (\phi(t_\gamma))_\gamma) \longrightarrow 0$ which implies that $(t_\gamma)_{\gamma\in\Gamma} \in \mathcal{A}_\xi$.

Likewise, we prove that if a τ_p-net has as τ_q-conets all the elements of $\mathcal{B}_\xi$, then it belongs to $\mathcal{A}_\xi$. Thus, ξ constitute a τ-cut in $(X,\mathcal{U})$.

It remains to prove that for every $i \in I$ (resp. for every $j \in J$), $(\xi^i_k)_{k\in K_i}$ (resp. $(\eta^j_\lambda)_{\lambda\in\Lambda_j}$) is $\tau(\overline{\mathcal{U}})$-convergent (resp. $\tau((\overline{\mathcal{U}})^{-1})$-convergent) to ξ. It is enough to prove it for $(\xi^i_k)_{k\in K_i}$.

We consider a τ_p-net $(\xi^i_k)_{k\in K} \in \mathcal{A}_{\xi^\star}$ and the τ-cut ξ in X the constructed above. If $(\xi^i_k)_{k\in K_i}$ is finally constant, then there exists $k_0 \in K$ such that $\xi^i_k = \xi^i_{k_0}$ for every $k > k_0$. But then, as in the case (a) of (A), we have $\mathcal{A}_{\xi^i_k} = \mathcal{A}_{\xi^i_{k_0}} \subseteq \mathcal{A}_\xi$ and thus $(\xi, \xi^i_k) \longrightarrow 0$.

If $(\xi^i_k)_{k\in K}$ is not finally constant, we have to examine the cases (b_1) and (b_2) of (A). In the first one, there exists $k_0 \in K$ such that for each $k > k_0$,

$\mathcal{A}_{\xi^i_k} \subseteq \mathcal{A}_{\xi^i_{k_0}} \subseteq \mathcal{A}_\xi$. Hence, $(\xi, \xi^i_k) \longrightarrow 0$. In the (b_2)-case, we can extract a subnet $(\xi^i_{k_\sigma})_{\sigma\in\Sigma}$ of $(\xi^i_k)_{k\in K_i}$ and a τ_p-net $(x_a)_{a\in A} \in \mathcal{A}_\xi$ such that $(\xi^i_{k_\sigma})_{\sigma\in\Sigma}$ and $(\phi(x_a))_{a\in A}$ are left cofinal. By the Proposition 20, $(\phi(x_a))_{a\in A}$ $\tau(\overline{\mathcal{U}})$-convergence to ξ. Therefore, by using the Proposition 15, we conclude that $(\xi^i_{k_\sigma})_{\sigma\in\Sigma}$ $\tau(\overline{\mathcal{U}})$-convergence to ξ. Finally, Proposition 13 and Proposition 15 imply that $(\xi, \xi^i_k) \longrightarrow 0$. This completes the proof. $\square$

The previous theorem implies the following theorem.

Theorem 28. Every T_0 quasi-uniform space has a τ-completion.

Lemma 29. If $(X,\mathcal{U})$ is T_0, so is $(\overline{X},\overline{\mathcal{U}})$.

Proof. Let $(\xi, \xi^\star) \in \bigcap\{\overline{U}\cap(\overline{U})^{-1} | U \in \mathcal{U}_0\}$. Suppose that $U \in \mathcal{U}_0$. There is a net $(x_a)_{a\in A} \in \mathcal{A}_\xi$ such that for each $(x^i_\beta)_{\beta\in B_i} \in \mathcal{A}_{\xi^\star}$ there holds $\tau.((x_a)_a, (x^i_\beta)_\beta) \in U$. By definition there is a $W \in \mathcal{U}_0$ such that $W^{-1}(x_a) \times W(x^i_\beta) \subseteq U$. Thus for each $(y^j_a)_{a\in A_j} \in \mathcal{B}_\xi$ we have that $\tau.(y^j_a, x^i_\beta) \in U$. Hence $\mathcal{B}_\xi \subseteq \mathcal{B}_{\xi^\star}$. Similarly from $(\xi, \xi^\star) \in \bigcap(\overline{U})^{-1}$ we conclude that $\mathcal{B}_{\xi^\star} \subseteq \mathcal{B}_\xi$. Thus $\xi = \xi^\star$. $\square$

We recall the classical definition of idempotency (adapted to the specific case of the τ-completion).

Definition 30. The τ-completion of a T_0 quasi-uniform space (X, d) is *idempotent* if and only if there exists a quasi-uniform isomorphism $\overline{\phi} : \overline{X} \longrightarrow \overline{\overline{X}}$ such that for each $\xi^\star \in \overline{X}$, we have $\overline{\phi}(\xi^\star) = \overline{\xi^{\star\star}}$ $(\xi^{\star\star} \in \overline{\overline{X}})$.

The *idempotency* of the τ-completion is established by the

Theorem 31. The τ-completion of a T_0 quasi-uniform space $(X,\mathcal{U})$ is idempotent.

Proof. We consider a τ-cut $\xi^{\star\star} \in \overline{\overline{X}}$ where $\xi^{\star\star} = (\mathcal{A}_{\xi^{\star\star}}, \mathcal{B}_{\xi^{\star\star}})$. Let $\overline{\phi}$ be the canonical embedding of $\overline{X}$ into $\overline{\overline{X}}$. Then, as in the Theorem 27, we define τ-cut $\xi^\star \in \overline{X}$ which is the end point of $\xi^{\star\star}$. Hence, $\xi^{\star\star} = \overline{\phi}(\xi^\star)$ which implies that $\overline{\overline{X}} = \overline{\phi}(\overline{X})$. On the other hand, by the aim of the Theorems 27 and 19 we can check that for each $U \in \mathcal{U}_0$, $(\xi^{\star\star}, \eta^{\star\star}) \in \overline{\overline{U}} \Leftrightarrow (\overline{\phi}(\xi^\star), \overline{\phi}(\eta^\star)) \in \overline{\overline{U}} \Leftrightarrow (\xi^\star, \eta^\star) \in \overline{U}$. The rest is obvious. $\square$

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Address: Athanasios Andrikopoulos
Department of Economics
University of Ioannina
Greece
E-mail address: aandriko@cc.uoi.gr

Address: John Stabakis
Department of Mathematics
University of Patras
Greece
E-mail address: jns@math.upatras.gr